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ON SOME FUNDAMENTALS OF PRE-CAMBRIAN PALEOGEOGRAPHY

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It is an established fact that the Archean basement complex (representing the Archeozoic era) has undergone not only complete metamorphism but also a world-wide intense folding which undoubtedly is a true folding due to world-wide diastrophism. The later Pre-Cambrian rocks (classed formerly as Huronian, then as Algonkian and more recently as Algoman, Huronian and Algonkian rocks, representing Proterozoic time) have escaped metamorphism and folding in the interior of North America but elsewhere have undergone like folding as the Archean basement complex and with few exceptions in the same sense; for example, in the eastern Canadian shield, where both are folded from the southeast (Adams). This world-wide folding of the Pre-Cambrian rocks stands in striking contrast to the localized folding of the earth crust in all later time.

Starting from this fact of the complete folding of the exposed Archean areas and on the proper inference that the entire Archean basement complex, and with it the greatest part of the Proterozoic rocks of the Earth, is likewise folded, the writer has attempted a synthesis of the directive lines of arrangement of these Pre-Cambrian folds.

In such an attempt there are to be excluded:

1. All areas of metamorphic rocks which are either proven to be or suspected of being younger than Pre-Cambrian age; such as are found in Greece, Asia Minor, the Andaman and Antillean Islands (serpentines of Cuba, etc.), the gneisses of the Coast range, lower California, etc.

2. All rocks of Pre-Cambrian age involved in Post-Cambrian folding, as those of North Africa, Spain, France, western Germany, the Alps, eastern Australia, etc. But it is to be noted here that even in these cases, keen observers have often enough found that the Pre-Cambrian nuclei of mountain ranges retain an independent original direction of folding, and further that

in some cases the new folds have clearly followed old lines of folding (post-humous folds). There are further left, even in those regions that were overrun by the crustal waves of Post-Cambrian time, 'islands' or blocks that remained undisturbed and that give important information on the original direction of folding in the Pre-Cambrian basement complex. Such undisturbed blocks are found in the Rocky Mountains, in Bohemia, on Borneo, in Cambodia, etc.

The large areas that are then left and lend themselves to our inquiry, are the greater part of North America, small regions of South America, eastern Europe, northern and eastern Asia, all inner Africa and western Australia. The general facts gleaned from them are complemented by those obtained from the unfolded 'islands' or blocks.

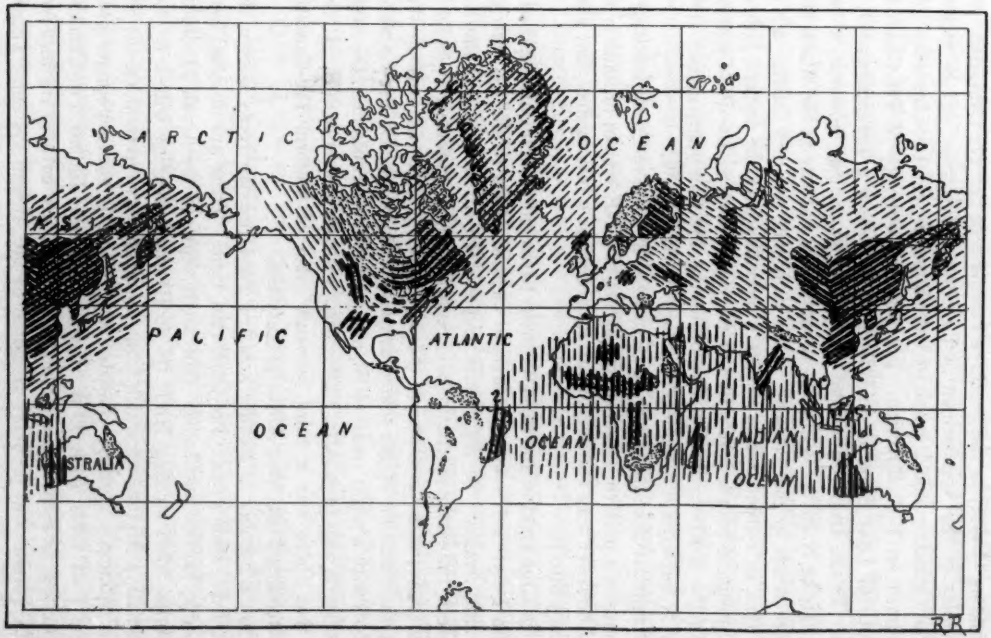
The most important of these areas are the so-called continental nuclei of North America (Canadian shield), Europe (Baltic shield) and Asia (Angara shield). An intended comparison of their principal directions of folding presupposes essential contemporaneity of the diastrophic revolutions they have undergone. This has been established by the correlations carried out by Adams, Sederholm and Willis. It is also to be noted that evidence of a like succession of eras and diastrophic events has been found in most of the other regions here mentioned as being amenable to our inquiry.

The source for this inquiry is chiefly Edward Suess' The Face of the Earth, where there is not only a large store of information as to the directions of Pre-Cambrian rock folding but where also the facts of the general directions of Pre-Cambrian folding in Asia and eastern Europe are clearly set forth. Nevertheless the assembling of the full data on Pre-Cambrian folding, which will be given in a later publication, requires laborious search through a widely extended literature.

With these introductory remarks we may briefly survey the preliminary results (which undoubtedly are still subject to important corrections) as to the arrangement of Pre-Cambrian folds.

In China, Richthofen, Pumpelly and Willis have found a uniform NE direction in the basement complex, which was completely folded and again abraded before the Cambrian submergence. This NE direction (the Sinian system of Richthofen and Pumpelly) continues through Korea, and Suess has shown its complete domination in all northeastern Asia east of Lake Baikal (his 'Baikal direction'), which is ENE-NE. This meets, in a meridional line, the 'Sajan direction' of Pre-Cambrian folds, which is WNW to NW and prevails west of Lake Baikal through all northern and middle Asia, Russia (Karpinski) and the Baltic shield, the latter being only a larger northern exposure of the Russian Pre-Cambrian plate. The latter extends through northern Europe much disturbed by later folding and probably abuts against the Laurentian plate in western Scotland.

In North America the Canadian shield exhibits in its eastern portion a uniform NE direction of the folds (Adams and Coleman). This direction


ARCTIC OCEAN
PACIFIC OCEAN
ATLANTIC OCEAN
INDIAN OCEAN
AUSTRALIA
R.R.
PRE-CAMBRIAN ROCKS
OBSERVED DIRECTIONS OF FOLDS
INFERRED DIRECTIONS

continues through Greenland with some variation to NNW along the coast of Labrador through Tertiary crustal movements (Adams) and it is even recognizable in the western edge of Scotland. It is further observed in several ways, as in the interlocking NE-SW bands of different gneisses, showing ancient folding (Keith) in the southern extension of the Pre-Cambrian basement complex in the Appalachian and Piedmont systems. In the western part of the Canadian shield, north of Lake Superior, the Pre-Cambrian folds have assumed an E-W direction. Likewise the outcrops of Pre-Cambrian rocks south of Lake Superior in Wisconsin etc. show a predominant E-W direction. In the Ozark Mountain Pre-Cambrian nucleus however, a NW direction (strike N 50°W) is already recognizable; likewise in the Black Hills and farther south in the Arbuckle uplift in Indian territory (strike WNW) and in Texas, in Burnet county; where a monadnock of the ancient Laurentian land projects with a NW strike of the major folds in the Pre-Cambrian rocks. In the southern Rocky Mountains the Pre-Cambrian masses strike from S to N but exhibit a tendency to turn NNW to NW.

It thus appears that there can be recognized in the Pre-Cambrian base of North America a uniform structure consisting of a NE system of folds in the east, that however does not abut, as in Asia, against a NW system, but gradually swings into it.

In Africa large areas of the Pre-Cambrian basement complex are exposed in the Sahara Desert and south, throughout Central Africa to the Red Sea and across it into Arabia. In the Sahara and western Africa as far as the Kalahari Desert, the strike of the folds is reported to be uniformly N-S, as far as known today, with variations in the east to NNW etc. Likewise in West Australia the direction of the folded areas of metamorphosed Pre-Cambrian rocks is N-S with a slight trend to W; and, as far as we could find, the northerly direction is also present in Madagascar with variations to NE. It thus appears that there was a third immense tract of uniform Pre-Cambrian folding extending from West Africa to Australia.

There are a number of smaller areas of Pre-Cambrian rocks known that have not been folded since Pre-Cambrian time. These are in Asia, found in Cambodia, Borneo and India. The latter shows in the Arvali Mountains Pre-Cambrian folds striking NNE (N 36°E) and extending within 65 Km. of the Himalaya Mountains which from the north have overridden the Pre-Cambrian Mountain system. Farther south there is also a younger group (Algonkian?) of folds with E-W direction. These Indian Pre-Cambrian folds, especially as seen in the Arvali Mountains, are entirely distinct from the Asiatic system and probably referable to the African system. There is likewise in Bohemia a Pre-Cambrian complex, striking NE for the most part, that is independent of the Russian plate and the Baikal direction. In America the Colorado mass, exposed in the Grand Canyon and south, represents a body of Archean and Algonkian folds with a NE direction and distinctly outside of the Laurentian system of North America.

In South America, NW to W striking folds in rocks of Archean aspect north of the Amazonas and in middle and western Brazil are all of younger than Cambrian age. It is however possible that the N-S (NNE) direction of the folds of Archean rocks in the Sierra do Mar and S. de Mantigueira in eastern Brazil can be correlated with the N-S folds in Africa and Australia. In Argentina, south of Buenos Ayres, mountain ranges composed of granite and gneiss with a SE strike represent, according to Suess, a continuation of the Brazilian basement complex.

The Antarctic Continent has afforded rocks of Archean aspect and may represent another shield; so far, however, these rocks have been found only in loose blocks.

The most important fact standing out from these data is that of the existence of at least three vast areas of Pre-Cambrian rocks of a supercontinental order of size, with uniform structure. One of these comprises Eurasia, exclusive of India and parts of northern and middle Europe; another Africa, West Australia and possibly India and eastern Brazil; the third North America with Greenland and the North Atlantic region to the Shetland Islands, but probably exclusive of the Colorado nucleus.

It will be a matter of further inquiry whether these wide tracts of supercontinental size that responded as units to the diastrophic forces that mark the world-wide Archeozoic and Algoman revolutions are to be considered as the first expressions of the gigantic continents that we find in Paleozoic time in the sense that they are ancestral or arch-continents. Indeed they well correspond to the later continents, as seen in the north Atlantic extension of North America across Greenland towards Europe; the Afro-Australian continent (Gondwanaland) and the Eurasian continental mass of Paleozoic times (see paleogeographic maps by Lapparent, Frech, etc.); the oceans that now separate the continents having arisen by a gradual collapse of the earth crust, probably along great circles.

There is, however, this further possibility to be considered; that the NE and SE Pre-Cambrian fold systems of Eurasia and North America form one belt and the N-S fold system of the equatorial regions in Africa and Australia another one. In that case we should have, on the northern hemisphere, surrounding the North Pole, a continuous belt that consists of two pairs of directions, each composed of a NE and NW striking system of folds; the whole forming a zigzagged belt; and another belt of N-S folds approximately following the equator. Between the two belts lie the independent blocks of India, Bohemia, Colorado, etc., all, or most of them with a NE direction as if, in some way, leading from the equatorial belt up to the circumpolar belt. It will probably have to be left to future observations whether there are indications of a corresponding circumpolar belt on the southern hemisphere as slightly suggested by the occurrence in Argentina.

If these belts actually exist as sketched here (see chart), then the question arises whether they do not represent cosmic agencies that influenced the

whole earth crust. There come to mind the suggestions of George Darwin as to the possibility of tidal waves in the earth crust as orogenic forces that would cause an equatorial N-S direction in the folds and a NE direction in the northern hemisphere; and those of Douvillé and Prinz who would explain, one the prevailing E-W, the other the N-S directions of many of the mountain ranges of post-Proterozoic time by a former greater velocity of revolution (Suess). It would seem that these belts of Pre-Cambrian folds lend themselves still more readily to an explanation by one or the other of these factors than do the post-Proterozoic more local fold systems. Since the Pre-Cambrian folding both in America and Eurasia has a southerly component (thrust from SE and SW), a retardation of the revolution of the Earth and a resulting wandering of the crust towards the poles seems to be indicated.

On the other hand, it must also be asked whether the world-wide folding of the Archean basement complex could not be explained by simply terrestrial forces. In this connection the result of close mapping of the Pre-Cambrian folds, carried out in late years in Bohemia and Scandinavia is of great importance. It brings out closely compressed folds, whose strikes are tortuous and wavy curves and often subcircular and even angularly broken lines. This, it has been concluded, points to a tangential pressure, acting from all sides on an earth crust of fairly uniform composition (Uhlich), a pressure and a composition that could be found only in Pre-Cambrian or rather Archean time, and that means a uniform contraction of the entire earth crust such as could not be invoked for the post-Proterozoic mountain systems. If this view should supply a competent explanation for the world-wide Archean folding, it still leaves unaccounted for the presence of large systems of uniform folding, which, as we have already seen suggests the view that in Proterozoic and even in Archeozoic time the crust was separated into masses that correspond in position if not in area and configuration to the continents of Paleozoic and more recent time.

THE INORGANIC CONSTITUENTS OF LOBSTER SHELLS

By Frank Wigglesworth Clarke and George Steiger

U. S. Geological Survey, Washington

Read before the Academy, November 18, 1918

In the course of an extensive investigation relative to the inorganic constituents of marine invertebrates, by Clarke and Wheeler,[1] it was found that among the distinctly magnesian organisms the proportion of magnesia was dependent upon the temperature of the water in which the animals live. The cold water animals contained much less magnesium carbonate than those

from warm waters; a relation which was strikingly manifest in the analyses of echinoderms and alcyonarians and which has been amply verified by a considerable number of new analyses made since the original memoir was published. In other groups of organisms the same relation was suggested, but not actually proved to hold, for there were exceptions that needed explanation. In a series of eleven analyses of crustaceans (crabs, lobsters, shrimps etc.), the same variation in magnesia was strongly indicated, but with irregularities which appeared to require further investigation. It was conceivable that different parts of a shell or skeleton might differ in composition, or else that variations might be due to differences in age. It had already been found in the case of two sea urchins that the spines contained much less magnesia than the main body of the shells, but the question relative to age remained to be investigated.

Through the kindness of Dr. H. M. Smith, director of the U. S. Bureau of Fisheries, the large claws of two lobsters (*Homarus americanus*) from a single locality, Boothbay Harbor, Maine, were obtained, one from a small lobster, the other from a large specimen. The analyses gave the following results, after rejecting organic matter and water and recalculating to 100 %

	SMALL LOBSTER	LARGE LOBSTER
$SiO_2 + (Al, Fe)_2O_3$	0.19	0.81
$MgCO_3$	6.02	11.51
$CaCO_3$	80.52	64.37
$CaSO_4$	1.29	1.85
$Ca_3P_2O_8$	11.98	21.46
	100.00	100.00

The difference between these two analyses is very great, the large animal being much more highly magnesian and phosphatic than the small one. Unfortunately, however, the actual sizes of the two lobsters were not given, and more precise data were evidently desirable. Accordingly Dr. Smith had fragments from three lobsters sent to us, all from the same station as the others, with definite figures as to length and weight. The fragments, moreover, in each case represented both the large claw and the carapace, so that variations in the individual as well as variations in age could be determined. The analyses, six in number, were as follows:

1. Small lobster, length 8½ inches, weight 10 ounces.
2. Medium lobster, length 11½ inches, weight 2 pounds.
3. Large lobster, length 16½ inches, weight 5½ pounds. The claw is indicated by a, the carapace by b.

	1a	1b	2a	2b	3a	3b
$SiO_2+(Al,Fe)_2O_3$	0.33	0.35	0.36	0.66	0.31	0.57
$MgCO_3$	10.81	7.74	11.28	8.12	10.99	8.77
$CaCO_3$	72.41	78.98	55.46	70.58	56.89	65.14
$CaSO_4$	1.24	1.23	2.12	1.58	2.32	2.32
$Ca_3P_2O_8$	15.21	11.70	30.78	19.06	29.49	23.20
	100.00	100.00	100.00	100.00	100.00	100.00

In each case the claw is richer in magnesium carbonate and calcium phosphate than the carapace. The variations due to age appear more distinctly when the average of each pair of analyses is taken, as follows:

	1	2	3
$MgCO_3$	9.27	9.70	9.88
$CaCO_3$	75.69	68.02	61.01
$Ca_3P_2O_8$	13.45	24.92	26.35
$CaSO_4$	1.24	1.85	2.32

Here the progressive increase in magnesium carbonate and calcium phosphate is clearly shown; and it also appears in the percentages of calcium sulphate, although the last detail is less significant. The smallest lobster, moreover, differs in the composition of its inorganic portion from that of the two larger animals much more than they do from each other.

From the evidence now at hand it seems clear that some of the departures from regularity in the proportions of magnesium carbonate in the shells or skeletons of marine invertebrates are due to one or both of the two causes which were suggested at the beginning of this paper. It is desirable, therefore, in further investigations of this kind, that in the study of the more highly specialized organisms the analyses should represent the totality of the inorganic portions, and that animals of the same degree of maturity should be taken. With the lower classes of organisms the difficulties are not so great, and regularities are much more easily discovered.

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[1] Clarke and Wheeler, *Prof. Paper*, No. 102, *U. S. Geological Survey, Washington.*

FLAGELLATE AFFINITIES OF TRICHONYMPHA

By Charles Atwood Kofoid and Olive Swezy

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The methods of division among the Protozoa are of fundamental significance from an evolutionary standpoint. Unlike the Metazoa which present, as a whole, only minor variations in this process in the different taxonomic groups and in the many different types of cells in the body, the Protozoa have evolved many and widely diverse types of mitotic phenomena, which are characteristic of the groups into which the phylum is divided. Some striking confirmation of the value of this as a clue to relationships has been found in recent work along these lines. The genus *Trichonympha* has, since its discovery in 1877 by Leidy,[1] been placed, on the one hand, in the ciliates and, on the other, in the flagellates, and of late in an intermediate position between these two classes, by different investigators. Certain points in its structure would seem to justify each of these assignments. A more critical study of its morphology and especially of its methods of division, however, definitely place it in the flagellates near the Polymastigina.

At first glance *Trichonympha* would undoubtedly be called a ciliate. The body is covered for about two-thirds of its surface with a thick coating of cilia or flagella of varying lengths, which stream out behind the body. It also has a thick, highly differentiated ectoplasm which contains an alveolar layer as well as a complex system of myonemes. The nucleus, however, is that of a typical flagellate. The flagella may equally well be called cilia, since they are arranged in longitudinal rows on the surface of the body, each arising from a minute basal granule imbedded in the ectoplasmic layer. Each basal granule, however, is connected with a fine fibril arising from the myonemes in the ectoplasm. The myonemes form a closely anastomosing network over the body, taking their origin from a complex structure at the anterior end which we call the centroblepharoplast (fig. 1, *c*). This corresponds to the blepharoplast of *Trichomonas* (fig. 6, *c*). The entire group of flagella are thus bound together into one integrated unit, the basis of which is the centroblepharoplast, forming the neuromotor system.

This integrated organelle system is found in a simple form in *Trichomonas* (fig. 6) where it consists of a centrosome-blepharoplast (*c*) imbedded in the anterior end of the axostyle (*ax*). From it arise the flagella and the undulating membrane with its marginal filament and parabasal body.

A more complex form of the same system and one which in some features leads toward the stage attained by *Trichonympha* with its much greater multiplicity of flagella, is shown in the motor organelle complex of *Giardia*.[2]

Here we find eight distinct flagella arising from different points on the surface of the body, but connected internally by a number of fibers which all take their origin from the centroblepharoplast complex attached to the anterior ends of the axostyles. The condition in *Giardia* differs from that in *Trichonympha* mainly in the small number of parts of its motor organelle complex. Correlated with the very great increase in the number of flagella in the latter species, is the vast increase in the number of distinct myonemes which have been developed. This has also necessitated an enlargement and increased complexity in the centroblepharoplast complex at the anterior end of the body. These structures still remain, however, distinctly flagellate in their structural relationships, though superficially resembling the ciliary coating found in the holotrichous Ciliata.

The mode of division found in *Trichonympha* offers still more striking proof of its flagellate affinities, and also emphasizes the wide divergence existing between it and the ciliates. As the name signifies the centroblepharoplast complex takes the rôle of centrosome in mitosis. In the prophase this entire structure divides by longitudinal splitting, followed by a division of the ectoplasmic part of the body, including a separation of the myonemes and flagella into two parts (fig. 2). As the centroblepharoplast divides the two moieties spin out a darkly staining band between them, the paradesmose (*par*).

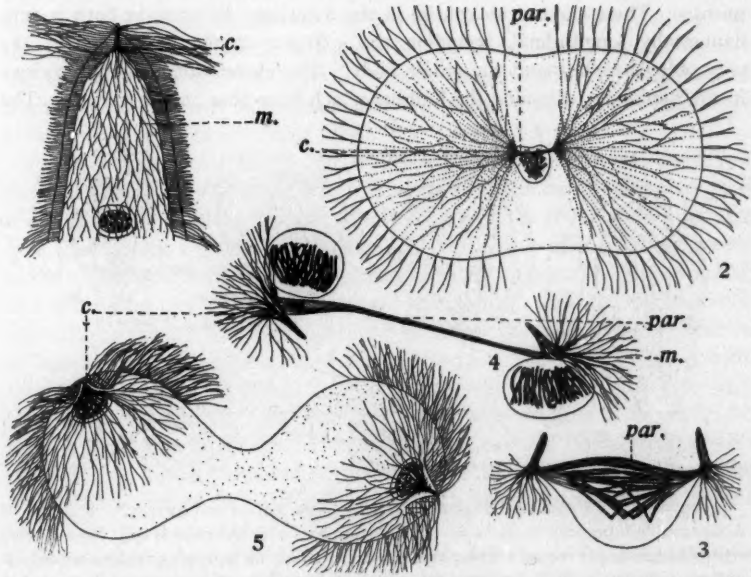
The nucleus migrates from its submedian position to the anterior part of the body. A precocious splitting of the chromosomes has taken place during the vegetative phase and the fifty-two chromosomes thus formed are reunited, forming twenty-six V-shaped chromosomes composed of distinct chromomeres (fig. 2). As the nucleus reaches the paradesmose it elongates until its length coincides with that of the paradesmose. The nuclear membrane remains intact throughout the entire process of division. Spindle fibers are formed from the ends of the paradesmose, or the centroblepharaplasts, pass through the nuclear membrane and become attached to the chromosomes, a fiber from one pole to one end while the other is connected with the opposite pole. With the shortening of the spindle fibers the chromosomes are straightened out in the equatorial plate stage (fig. 3).

In all these mitotic figures the divided centroblepharoplast functions as the centrosome, while the myonemes, surface ridges and flagella are found attached to each half and streaming out from it like the astral rays in mitosis, in a metazoan nucleus. The paradesmose lies above the nucleus, that is, towards the surface of the body, outside the nuclear membrane but partly enfolded within it.

In the telophase the constriction of the nuclear membrane takes place with the chromosomes still showing their attachment to the spindle fibers. The chromosomes are never drawn close to the poles as in mitosis in *Trichomonas*, but the fibers fade out before reconstruction begins in the nucleus

(fig. 4). As the daughter nuclei round up they become free from the paradesmose which still connects the two daughter blepharoplasts (fig. 4). Later the paradesmose loses its staining reactions and fades out or is absorbed, either by the cytoplasm or by the centroblepharoplasts. This is followed by division of the body and separation of the two daughter flagellates.

A comparison of various points of this process with certain ones in *Trichomonas* reveals a striking similarity in the two types. In the latter the

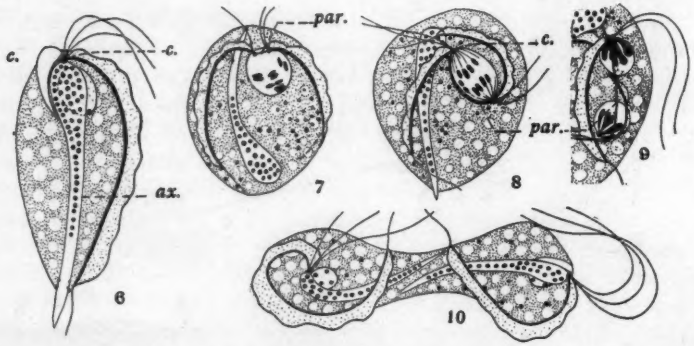


FIGS. 1–5. *Trichonympha campanula* Kofoid and Swezy. *1.* Anterior portion of trophozoite showing three groups of flagella, centroblepharoplast, myonemes and uncleus. × 225. *2.* Late prophase of division. Ectoplasmic structures and flagella divided; paradesmose connecting the centroblepharoplasts; 26 split chromosomes. ×225. *3.* Metaphase of nucleus, × 600. *4.* Telophase. Centroblepharoplasts still connected by paradesmose. × 600. *5.* Mitosis completed, cytoplasmic division approaching. × 225.

division of the centroblepharoplast also produces a connecting paradesmose (fig. 7, *par.*). Attached to each new centrosome or centroblepharoplast are the motor organelles, which are produced partly by division of the old structures and partly by new outgrowths. The condition here is similar to that in *Trichonympha* (fig. 2). A precocious splitting of the chromosomes produces ten moieties which become reunited at the time of spindle formation, giving five as the number of chromosomes going to each daughter cell. The same process also occurs in *Trichonympha.*

The spindle fibers are formed from the ends of the paradesmose or the centroblepharoplasts, pass through the nuclear membrane, which remains intact, and the chromosomes become arranged upon the spindle in the equatorial plate by a process analogous to that found in *Trichonympha*. The paradesmose is found outside the nuclear membrane (figs. 8, 9) and fades out in the late telophase precisely as does that of the other species.

It is thus evident that these two organisms, though differing widely in their general morphological characters, yet follow precisely similar modes of mitosis. These similarities consist in the following; Division in both is fundamentally longitudinal, including the splitting of the chromosomes, the centroblepharoplast and the entire body. The chromosomes are precocious in their splitting, doubling the number which later goes on the spindle. The



FIGS. 6–10. *Trichomonas angusta* Alexeiefft, after Kofoid and Swezey.[4] × 1500. *6* Active motile form prior to division. *7*. Late prophase with five pairs of split chromosomes centroblepharoplasts connected by paradesmose. *8*. Late metaphase; paradesmose outside nuclear membrane. *9*. Telophase with centroblepharoplasts still connected by paradesmose. *10*. Mitosis completed, cytoplastic division approaching. *Ax.*, axostyle; *c.*, centroblepharoplasts; *m.*, myonemes; *par.*, paradesmose.

centroblepharoplast divides with a paradesmose formed to connect the two parts as they separate. The motor organelles are divided with new outgrowths to complete the full complement of each daughter cell. These remain attached to the centroblepharoplasts throughout the entire process of division. The nuclear membrane persists, with the spindle fibers passing through from the ends of the paradesmose, which remains outside the membrane. Pseudosynapsis occurs as part of the chromosome cycle in both forms. The paradesmose persists through the late telophase stage after the rounding up of the daughter nuclei, fading out before plasmotomy occurs.

Two phases of these processes deserve particular attention. These are the formation of the paradesmose and the attachment of the motor organelles to the centrosomes. So far as present records go both of these phenomena are

typically and solely flagellate in their occurrence. Nothing comparable is known to occur in the ciliates in which at mitosis the cilia bear no morphological relation to the centrosomes or to any other part of the mitotic figure.

In the group of flagellates comprised in the order Polymastigina, the paradesmose is a characteristic feature of mitosis in so far as these processes have been studied. In the trichomonad flagellates, as pointed out above, its occurrence is constant and characteristic. In the nearly related form, *Hexamitus*, the paradesmose is also present.[3] A peculiar development of this structure is found in the so-called sphere of *Noctiluca*. This is produced by the elongation and division of a mass of differentiated archoplasm outside the nucleus. The exact relation of this to the tentacle and flagellum of *Noctiluca* has not been determined, but its position would suggest that some close connection between them exists.

It has been pointed out in our earlier paper[4] on trichomonad flagellates that the paradesmose is not a precise homologue of the centrodesmose or central spindle of the metazoan type of mitosis. It is the result of two peculiar specializations in trichomonad division, the continuity of the nuclear membrane which excludes it from the typically axial position of the central spindle, and the connection of the centrosome-blepharoplast with the entire motor organelle complex. The latter feature is probably the most important one in its bearing on the development of the paradesmose, the strain on the spindle resulting from the constant activity of a group of flagella attached to each pole, requiring the development of a stronger support than is found in the mitotic structures where this condition does not prevail. As a result of this necessity we find a paradesmose developed in those flagellates where the flagella form part of the mitotic figure, but it is usually, if not always, absent where this does not occur.

In no other group of Protozoa do we find a structure approximating the paradesmose of the flagellates. Among the ciliates division of the nucleus occurs as distinct phenomena separated from, though synchronous with, the division of the motor organelles and other structures of the body.

Another point in the division process of *Trichonympha* which is equally important as those mentioned above, is the fact that division is longitudinal. This is shown in the longitudinal splitting of the centroblepharoplast and the ectoplasmic structures. This fact also serves to separate it from the ciliates where transverse division is general, and allies it with the flagellates.

The occurrence of these peculiar flagellate specializations in *Trichonympha* would therefore preclude the possibility of any ciliate affinities for that genus. This conclusion receives confirmation from a careful analysis of its morphological features which, though superficially ciliate in appearance, yet are fundamentally flagellate in their character and relationships.

The relations of other members of the group of curious and peculiar organisms which, in company with *Trichonympha*, are parasitic in the intestinal

tract of the termites, have also been obscured by the high degree of specialization shown in their motor organelles. In one of these, *Joenia*, Grassi and Foa,[5] have figured longitudinal division with the formation of a prominent paradesmose which persists until the daughter cells are ready for plasmotomy. The paradesmose here, as in other flagellates, is intimately connected with the flagella and their related neuromotor structures.

In *Lophomonas* the old motor organelles disappear, according to Janicki[6] who has described division in this form. An entire new motor organelle system is developed from the ends of the paradesmose after completion of division of the nucleus. The exact origin of the paradesmose in this case cannot be determined from the figures of Janicki, but it arises from the nuclear-neuromotor complex. It remains outside the nuclear membrane, hence is evidently a true paradesmose.

In the remainder of this group mitotic phenomena have not yet been described and it is in their morphology only that we must look for relationships. These are found in the relations of flagella, internal myonemes and centroblepharoplasts with their various modifications, the neuromotor system. In *Spirotrichonympha* Grassi[7] has figured, in these structures, relations which are comparable with those found in *Trichonympha*, although much simpler. The flagella are arranged in spiral courses around the body starting from the anterior tip. Each series is accompanied by a slender band or myoneme extending along the line of basal granules beneath the surface of the body. The number of lines of flagella and myonemes varies slightly in the different species. At the anterior tip of the body these are joined in a small granular mass, the centroblepharoplast.

Zulueta[8] has figured the same spiral myonemes in *Dinenympha* without the series of flagella outlining their course. A single flagellum arises from the end of each myoneme. At the time of division the small granular mass at the anterior tip of the body divides, each moiety taking four of the myonemes, and forming a paradesmose between them as they separate. This centroblepharoplast acts as the centrosome in the formation of the spindle for the division of the nucleus, with its attached myonemes taking the place of the astral rays, as in *Trichonympha* (fig. 2).

The same relations of motor organelles and internal myonemes, by means of which an integrated neuromotor system is formed, may be found throughout all the members of this group of organisms. The range in complexity extends from the trichomonad type of structure to that shown in *Trichonympha*, which exhibits a higher degree of specialization and development than do many of the lower Metazoa. This specialization is confined almost exclusively to the motor organelles and the accessory structures connected with them, the neuromotor system. At the time of division this acts as a unit, dividing and half going to each daughter cell. An apparent exception to this is found in *Lophomonas*, but here a part of the old neuromotor system

is evidently retained in the paradesmose and from it the new motor organelles of each daughter cell are formed.

In these structures and more particularly in the processes by means of which they are passed from one generation to the next, we have a clue to relationships that seems to be more fundamental and significant in its importance than are external features, which are constantly subject to modifications through environmental changes. The amount of specialization has in no way changed the basic facts of these relations and processes. Thus in the highly complex *Trichonympha* there are precise homologues with the simpler features in *Trichomonas* with its simple centroblepharoplast granule and few flagella.

The superficial resemblance between the trichonymph parasites of the termites and the ciliates is the result of the high degree of specialization and evolutionary development to which the former have attained, and is not indicative of a derivative relationship of even the most remote kind between the two. The Trichonymphidae are fundamentally and characteristically flagellate in their type of structure as well as in their methods of division. We may therefore dismiss completely the early conception of Leidy, Kent and their followers that the Trichonymphs are ciliates and revise our wide-spread conception that flagella are universally or even characteristically few in number. These protoplasmic processes are flagella primarily because of their relation to the nucleus and the mitotic figure. Flagella are attached directly or indirectly to the centrosome and share in mitotic processes. Cilia are not thus attached and have no correlated part in mitosis. Numbers contribute no necessary part of this definition of flagella, they apparently do of cilia.

We may also dismiss the later conception of Hartmann[3] that the Trichonymphs are an intermediate group between flagellates and ciliates. In the first place a transition type between these primary groups can not be expected as parasites of a highly organized group of social insects. The appearance of transition is illusory, depending on superficial structures and numbers merely, while the deeply significant mitotic process and its structures are unequivocally flagellate in nature. We therefore reject Hartmann's[3] transitional conception and with it his Trichonymphida and retain Grassi's Hypermastigina as the fitting as well as the legitimate designation for this most highly specialized group of the flagellates.

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[6] Janicki, C., *Zs. wiss. Zool., Leipzig*, **95**, 1910 (243–315), pls. 6–9.

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[9] Hartmann, M. *Hertwig's Festsch.*, **1**, 1910 (349–392), pls. 27–30, 3 figs, in text.

THE TERNARY SYSTEM CaO-MgO-SiO_2

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A number of investigations dealing with one or more of the four oxides, lime, alumina, magnesia and silica, have in recent years been carried out at the Geophysical Laboratory of the Carnegie Institution of Washington, as a preliminary step in the study of the rocks and minerals of the earth's crust. These studies[1] include all of the possible binary systems and three of the four possible ternary systems which may be constructed from these oxides. The fourth ternary system has been studied only in part. We desire in this paper to present a summary of the results we have obtained in a study of the remainder of this fourth ternary system, and also to correlate these results with the results previously obtained.

The experimental methods employed are similar to the methods used in the previous investigations at this laboratory. Samples of known compositions were prepared by fusing together in platinum crucibles weighed amounts of chemically pure calcium carbonate, magnesia and silica and subsequently reducing the samples to fine powders. The production of homogeneous samples usually necessitated the repeating of this process once or twice. The investigation of a sample was conducted as follows: a small quantity of the sample was wrapped in a piece of platinum foil which was about 1 sq. cm. in area and this charge was tied by a fine platinum wire to a small ring of marquardt porcelain. The porcelain ring was hung on a fine platinum wire, the ends of which were connected to two stout platinum leads. A marquardt porcelain tube carried these leads and also a platinum-platinrhodium thermoelement the junction of which was not more than a few millimeters from the charge. The charge, thermoelement, etc., were inserted into a hot platinum resistance furnace and the furnace temperature was regulated in such a manner as to maintain the charge at a predetermined temperature for a sufficient time to allow an equilibrium condition to be attained (usually from fifteen to thirty minutes but sometimes as long as forty-eight hours). The wire supporting the ring and the charge was then fused by passing an electric cur-

rent through it, and the charge fell into a mercury bath which instantly chilled it. When cold the charge was opened and the content was examined under a petrographic microscope by means of which the phases present could be identified.

The results of the hundreds of heat treatments which were carried out during this study cannot be presented in detail in a paper as brief as this one must be, but we hope in a general way to indicate their character.

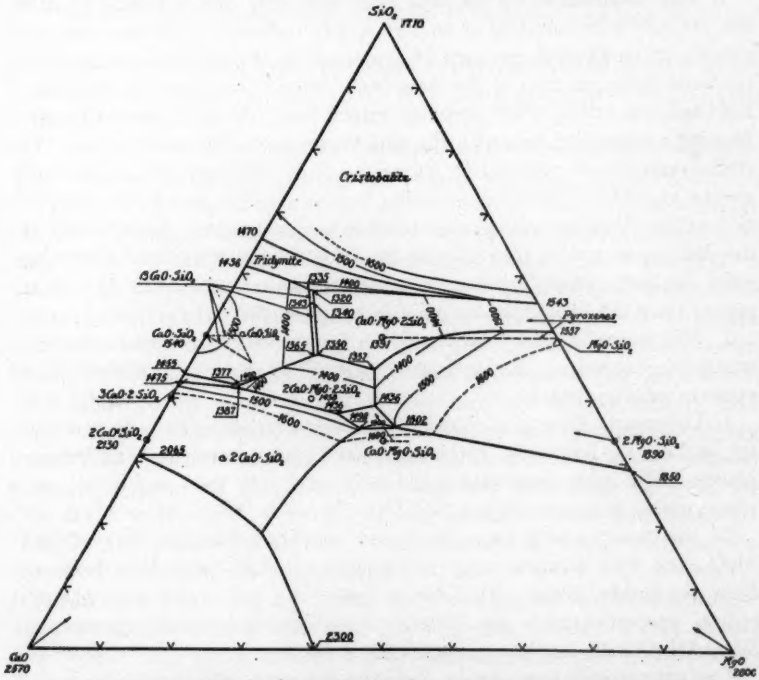

FIG. 1. THE TRIANGULAR CONCENTRATION DIAGRAM IN MOL-PER CENT

The heavy lines are the boundaries of the fields, the solid light lines are the isotherms which are readily measured, the lighter dotted lines are the high temperature isotherms which are not so accurately known and the heavier dotted lines are limits of the solid solutions. The solid solutions may also be distinguished from the isotherms in that they have arrows which indicate the fields which correspond to the respective solid solutions. The temperatures given correspond to (1) the quintuple points, (2) the quadruple points, (3) the melting points of the compounds and (4) the melting points of the component oxides.

The phases which occur in the three main binary systems and which also have fields in the ternary system are: (1) Cristobalite, SiO_2, (2) Tridymite, SiO_2, (3) Pseudowollastonite, α $CaO.SiO_2$, (4) Tricalcium di-silicate, $3CaO.2SiO_2$, (5) α Calcium orthosilicate, α $2CaO.SiO_2$, (6) Lime, CaO, (7) Periclase, MgO, (8) Forsterite, $2MgO.SiO_2$.

The binary compounds which do not have fields in the ternary system are: (1) Tricalcium silicate, $3CaO.SiO_2$, (2) Clino-enstatite, $MgO.SiO_2$.

The ternary compounds which have fields in the ternary system are: (1) Diopside, $CaO.MgO.2SiO_2$, (2) $2CaO.MgO.2SiO_2$.

The additional phases which have fields in the ternary system are not of constant composition but are ternary in character. They are:

1. The wollastonite ($\beta CaO.SiO_2$) solid solutions. These consist of either two series of solid solutions or an area of solid solution. The one series may contain up to 18 mols. per cent of diopside, $CaO.MgO.2SiO_2$ and the other up to 44 mols. per cent of the compound $2CaO.MgO.2SiO_2$. If an area of solid solution exists, it will probably extend from the one series to the other forming a triangular area with the apex at the $CaO.SiO_2$ composition. The wollastonite-pseudowollastonite inversion, $\beta CaO.SiO_2 \rightleftarrows \alpha CaO.SiO_2$ normally occurs at 1200°C. but this inversion temperature is raised by the addition of the dissolved substances reaching a maximum of 1343°C. with the diopside series and of 1365°C. with the $2CaO.MgO.2SiO_2$ series. Only those solid solutions which invert at the higher temperatures occur as primary phases since the liquidus at no point falls below 1320°C.

2. The pyroxene solid solutions which form a continuous series with end members, diopside, $CaO.MgO.2SiO_2$ and clino-enstatite, $MgO.SiO_2$, all occur as primary phases.

3. The monticellite ($CaO.MgO.SiO_2$) solid solutions which may contain up to 11% of forsterite, $2MgO.SiO_2$, are partly represented as primary phases. The pure compound monticellite probably does not occur, as a phase, stable in contact with a liquid.

Of the phases which have just been enumerated the $\beta CaO.SiO_2 - 2CaO.MgO.2SiO_2$ solid solutions, and the compound $2CaO.MgO.2SiO_2$ have not been previously noted. This latter compound has practically identical optical properties with the mineral åkermanite for which the formula $8CaO.4MgO.9SiO_2$ has been proposed, by Schaller.[2]

The summarized temperature and concentration relations are shown in figure 1. The compositions are herein represented on an equilateral triangle in mol-per cent, and the diagram includes (1) the location of the fields, with their boundary curves, (2) the location of the invariant points, (3) the temperatures which correspond to the fixed points, and (4) the isotherms which indicate the slopes which the various fields have on a solid model upon which the temperatures of complete melting are represented as vertical distances above a triangular concentration diagram similar to the one shown.

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QUANTITATIVE RELATIONS BETWEEN CHROMATIN AND CYTOPLASM IN THE GENUS ARCELLA, WITH THEIR RELATIONS TO EXTERNAL CHARACTERS

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A problem that has attracted considerable attention during the past twenty years is that of the quantitative relations between the nucleus and the cytoplasm of animal cells.

According to Richard Hertwig, who has been the foremost advocate of the nucleo-cytoplasmic-relation theory, a balance between nuclear and cytoplasmic masses exists in the normal cell, this balance being due to the interchange of materials between nucleus and cytoplasm. This state of equilibrium may be disturbed by such factors as changes in temperature, over-feeding or starvation. The result of these disturbing agents is an excess of nuclear over cytoplasmic materials. This excess of nuclear material leads to the depression of the cell, which finally ends in death unless normal mass relations are re-established in some way. The restoration of the normal equilibrium may be attained by the giving up of nuclear material to the cytoplasm, by ordinary cell division, or by the addition of a foreign element through conjugation. Hertwig and others have attempted to account for many of the complicated stages in the life cycles of Protozoa with this theory, but while this hypothesis is apparently applicable to many phenomena it will not bear close analysis, and a review of the extensive literature on this subject reveals a fatal lack of data on which to base the theory.

The material that I have used in my investigations consisted of several species of shelled fresh-water Protozoa of the genus Arcella. Arcella dentata has a shell with tooth-like projections extending out from the periphery. This shell averages about 130 microns in diameter and 50 microns in thickness.

There is a circular mouth opening in the ventral wall of the shell, through which protoplasmic extensions are pushed out that serve as locomotor organs and for capturing food. The cytoplasmic body within the shell contains two nuclei situated on opposite sides of the mouth opening. These nuclei are of the vesicular type, with the chromatin aggregated into a spherical mass in the center. Since the shell is almost transparent, especially in young specimens, it is easy to measure both the entire nucleus and the chromatin-mass within it, in the living animal. This makes Arcella peculiarly favorable for nucleo-cytoplasmic studies. The usual method of reproduction is binary division. When a certain amount of protoplasm has been built up within the shell and environmental conditions are favorable, the cytoplasm protrudes from the mouth of the shell and secretes a new shell; then, approximately one-

half of the protoplasm passes back into the old shell, and the two shells separate. It is thus possible to distinguish between parent and offspring.

The experiments performed on pure lines of Arcella dentata prove that a definite relation exists between nuclear number and cell size within each pure line.

One of the members of line 150 was cut in two so that each half contained a single nucleus. Both halves continued to live and reproduce; the offspring were slightly irregular in shape, but the spines could be counted easily and the diameter measured. These offspring were smaller than the original parent and possessed fewer spines. They were each provided with only one nucleus. From these two uninucleate half-specimens were derived 209 uninucleate descendants which had a mean spine number of 11 and a mean diameter of 116 microns.

After a number of generations that differed in different cases, all of the uninucleate specimens produced empty shells and became binucleate again. Apparently during this process the single nucleus divided into two, as in ordinary division, but the new shell that was formed, was cast off empty, and all of the cytoplasm and both nuclei were retained in the parent shell. The binucleate specimens thus formed, gave rise at once to binucleate offspring. These offspring were larger than the parent; and their offspring were still larger. In this way a gradual increase in diameter and in spine number took place generation after generation, until at the end of the third or fourth generation, an equilibrium was established and a mean diameter and spine number were regained characteristic of the line before the experiments were begun. Thus in line 150, the size of the organisms and the characteristics of the shell depend upon the number of nuclei, and each nucleus is accompanied by a rather definite quantity of cytoplasm.

Other lines of Arcella dentata were subjected to this and other kinds of operations and the data obtained fully confirm the conclusion just stated.

Arcella polypora differs from Arcella dentata in the absence of spines and in the greater number of nuclei. In this species the nuclei are distributed at approximately equal distances from one another. It was found that within a line derived by fission from a single specimen, the number of nuclei varied. In line 5 they varied from 3 to 7. When the diameters of these specimens were compared with the number of nuclei they contained, a remarkably high correlation was revealed. The mean diameter of specimens with 3 nuclei was 109 microns; of those with 4 nuclei, 113; of those with five nuclei, 120; of those with six nuclei, 127; and of those with seven nuclei, 130 microns. It is evident that as the number of nuclei increases the size of the organism increases, and that in this species as in Arcella dentata, a rather definite amount of cytoplasm accompanies each nucleus. No specimens with less than 3 nuclei appeared in the cultures, so operations were resorted to, in order to obtain individuals with one and two nuclei. Specimens were cut into

pieces, and these pieces continued to live and reproduce. From these pieces a few offspring with one and 2 nuclei were obtained. Specimens with one nucleus averaged 82 microns in diameter; and those with two nuclei, 86 microns. Here, as in normal specimens, size and nuclear number are closely correlated.

Other lines of Arcella polypora behaved in similar fashion, but the data show that the ratio between nuclear number and cell size differs markedly in the different lines. Thus the number of nuclei in line 34 ranged from 5 to 10, but the specimens were smaller than were those with a lesser number of nuclei in line 5. The specimens in line 5 with 4 nuclei were similar in size to those in line 34 with 8 nuclei; those in line 5 with 5 nuclei were about as large as those in line 34 with 9 nuclei; and those in line 5 with 6 nuclei, approximated in diameter those in line 34 with 10 nuclei.

This condition suggested the possibility that size in these organisms is controlled not by the number of nuclei, but by the *amount of chromatin* within the nuclei. Accordingly, measurements were made of the chromatin masses in specimen from a number of lines, and this hypothesis was confirmed in a remarkable manner.

Measurements were first obtained of specimens of Arcella dentata. It soon became evident that in this species, the larger the specimen the greater the amount of chromatin within the nuclei. A chromatin-cytoplasmic mass relation was thus established. Measurements were then made of the chromatin masses in specimens from lines 5 and 34 of Arcella polypora. From these measurements, the volumes of the chromatin masses were computed, and they were found to be greater in line 5 than in line 34. For example, the total volume of the 6 chromatin masses in an average specimen of line 5 proved to be approximately equal to the total volume of the 10 chromatin masses of an average specimen of the same size in line 34. Similar conditions were found to exist when the total volume of chromatin within other specimens of line 5 was compared with the total volume of chromatin within specimens of the same size in line 34. These data prove that the quantity of chromatin contained in specimens of the same size within the two lines, 5 and 34, was very nearly the same, regardless of the variations in the number of nuclei. Thus, in Arcella polypora, as in Arcella dentata, there is a certain amount of chromatin associated with a certain amount of cytoplasm, and a definite chromatin-cytoplasmic-mass-relation is shown to exist.

The data presented above not only show a mass relation between chromatin and cytoplasm, but also a relation between chromatin mass and external measurable characters. For example, in Arcella dentata, both the diameter of the shell and the number of spines are correlated with the chromatin mass; and in Arcella polypora, the diameter of the shell also depends upon the total volume of chromatin.

A summary of some selection work that I carried on last year with Arcella dentata was published in the October number of the PROCEEDINGS of this

Academy. I concluded from this work on selection, that from the descendants of a single specimen produced by binary division, lines could be distinguished that were hereditarily diverse as regards spine number and diameter. The present study indicates that these heritable diversities may have been due to changes in the volume of the chromatin. These chromatin-cytoplasmic studies also have a bearing on the selection work carried on by various investigators, notably by Jennings with Difflugia and by Root with Centropyxis. Arcella, Difflugia, and Centropyxis all belong to the same group of Protozoa, the fresh-water Rhizopods; but the nuclei can not be seen in either Difflugia or Centropyxis, and hence at least some of the results obtained by Jennings and by Root may have been due to changes in nuclear number and consequently in chromatin mass, rather than an hereditary change in the chromatin itself as seemed probable. An increase or decrease in nuclear number, however, does not account for simultaneous and independent diversities such as Jennings found in Difflugia with respect to shell diameter and length of spines, unless the assumption is made that certain nuclei exert an influence upon certain shell characters and other nuclei upon other shell characters. Evidence was obtained from my studies of Arcella polypora that hereditarily diverse strains with respect to nuclear number and shell diameter could be distinguished within a single line. More data regarding this and other related problems are very desirable.

A THEOREM ON POWER SERIES, WITH AN APPLICATION TO CONFORMAL MAPPING

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Note I on Conformal Mapping under aid of Grant No. 207 from the Bache Fund.

Theorem: Given a power series $w(z) = \sum_0^\infty a_n z^n$ *convergent for* $|z| < 1$ *and such that* $\sum_0^\infty n |a_n|^2$ *converges, and writing*

$$z = 1 - \rho e^{\theta i}, \quad z' = 1 - \rho e^{\theta' i},$$

where $\rho > 0$ *and* θ *and* θ' *vary with* ρ *subject only to the conditions*

$$-\frac{\pi}{2} + \epsilon \leqq \theta \leqq \frac{\pi}{2} - \epsilon, \quad -\frac{\pi}{2} + \epsilon \leqq \theta' \leqq \frac{\pi}{2} - \epsilon,$$

ϵ *being positive but arbitrarily small, then*

$$w(z) - w(z') \to 0 \text{ as } \rho \to 0$$

uniformly in respect to θ *and* θ'.

It should be noted that no assumption is made regarding the convergence of the power series at $z = 1$. For the proof, we write $w(z) = w_1(z) + w_2(z)$, where

$$w_1(z) = \sum_0^{N-1} a_n z^n, \quad w_2(z) = \sum_N^{\infty} a_n z^n.$$

First, we have

$$w_1(z) - w_1(z') = \sum_1^{N-1} a_n (z^n - z'^n)$$

$$= (z - z') \sum_1^{N-1} a_n (z^{n-1} + z^{n-2} z' + \cdots + z'^{n-1})$$

and since $|z - z'| = \rho |e^{\theta i} - e^{\theta' i}| \leqq 2\rho$, $|z| < 1$ and $|z'| < 1$, it follows that

$$|w_1(z) - w_1(z')| \leqq 2\rho \sum_1^{N-1} n |a_n|. \tag{1}$$

In the second place,

$$|w_2(z) - w_2(z')| \leqq \sum_N^{\infty} |a_n| \, |z^n - z'^n| = \sum_N^{\infty} \sqrt{n} |a_n| \cdot \frac{|z^n - z'^n|}{\sqrt{n}},$$

and applying Lagrange's inequality

$$|w_2(z) - w_2(z')|^2 \leqq \sum_N^{\infty} n |a_n|^2 \cdot \sum_N^{\infty} \frac{|z^n - z'^n|^2}{n}$$

$$\leqq \sum_N^{\infty} n |a_n|^2 \cdot \sum_1^{\infty} \frac{|z^n - z'^n|^2}{n}.$$

Denoting by the $\bar{z}$ the conjugate of z, we have

$$|z^n - z'^n|^2 = (z^n - z'^n)(\bar{z}^n - \bar{z}'^n) = (z\bar{z})^n + (z'\bar{z}')^n - (z\bar{z}')^n - (z'\bar{z})^n,$$

and consequently, since $|z| < 1$, $|z'| < 1$,

$$\sum_1^{\infty} \frac{|z^n - z'^n|^2}{n} = \log \frac{(1 - z\bar{z}')(1 - z'\bar{z})}{(1 - z\bar{z})(1 - z'\bar{z}')}.$$

Introducing $z = 1 - \rho e^{\theta i}$, $z' = 1 - \rho e^{\theta' i}$, a simple calculation shows that for $\rho < \sin \epsilon$ and observing the limitations governing θ and θ',

$$\frac{(1 - z\bar{z}')(1 - z'\bar{z})}{(1 - z\bar{z})(1 - z'\bar{z}')} = \frac{2 + 2\cos(\theta + \theta') - 2\rho(\cos\theta + \cos\theta') + \rho^2}{(2\cos\theta - \rho)(2\cos\theta' - \rho)}$$

$$< \frac{2 + 2 + 2(1 + 1) + 1}{(2\sin\epsilon - \sin\epsilon)(2\sin\epsilon - \sin\epsilon)} = \frac{9}{\sin^2\epsilon},$$

so that finally

$$|w_2(z) - w_2(z')| \leqq \left(\sum_N^{\infty} n |a_n|^2\right)^{\frac{1}{2}} \left(\log \frac{9}{\sin^2\epsilon}\right)^{\frac{1}{2}} \tag{2}$$

for $\rho < \sin \epsilon$.

Since $\Sigma n |a_n|^2$ converges, the right hand member in (2) may be made less than an arbitrarily small δ for $\rho < \sin \epsilon$ by taking N sufficiently large, and having fixed N, the right hand member in (1) may be made less than δ by taking ρ sufficiently small. Thus, for ρ sufficiently small, $|w(z) - w(z')| < 2\delta$ independently of θ and θ', which proves the theorem.

Let us now assume in particular that $w = w(z)$ maps the circle $|z| < 1$ conformally on a simple (i.e. simply connected and nowhere overlapping) region D in the w-plane, and that all points of D are within a circle of radius R (this latter condition can always be brought about by a linear transformation on w and the extraction of a square root [1]); then $\sum_1^\infty n |a_n|^2$ converges [2] (and is less than R^2). Finally, suppose that $w(z')$ approaches a limit w_0 when z' approaches unity on the real axis; our theorem then shows that $w(z)$ approaches the same point w_0 on the boundary of D when z tends toward unity along any curve interior to both the unit circle and an angle less than π formed by two straight lines through $z = 1$ and symmetrical in respect to the real axis. This proposition is usually derived, somewhat less directly, from the distortion theorem.[3]

[1] Koebe, *J. Math., Berlin*, 145, 1915.

[2] Fejér, *Festschrift . . . H. A. Schwarz*, Berlin, 1914.

[3] Koebe, l. c., and Study, *Konforme Abbildung einfach zusammenhängender Bereiche*, Leipzig, 1913.